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An internal wave in a viscous ocean stratified by both salt and heat

By N. H. THOMAS AND T. N. STEVENSON
Department of the Mechanics of Fluids, University of Manchester

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The propagation of an internal wave in an ocean which is stratified in both temperature and salinity is considered. It is shown how the effects of viscosity, heat conduction and solute diffusion attenuate a cross-wave from an oscillatory disturbance. Under certain conditions there is no wave solution even though the oscillatory frequency is less than the Brunt–Väisälä frequency and the fluid is stably stratified.

1. Introduction

When a body oscillates with a frequency ω in a density stratified fluid which has a constant natural frequency ω_0, the inviscid equations show that energy propagates along straight lines inclined at the angle $\sin^{-1}(\omega/\omega_0)$ to the horizontal, forming a cross-wave. Thomas & Stevenson (1972) discuss the effects of viscosity on this cross-wave. (The paper will be referred to as I.) It was demonstrated both theoretically and experimentally how viscosity increases the width of the wave and attenuates the velocities away from the forcing region. The experiments were in stratified brine but salt diffusion was neglected in the analysis.

In this note it is shown how heat conduction and solute diffusion modify the Boussinesq form of the equations in I. It is shown that the omission of salt diffusion in the original paper is justified.

The effect of gradients of two properties and oscillatory instabilities in particular have been considered by Veronis (1965, 1968) and Walin (1964) in convection flows. It is found that the present internal-wave solution breaks down under the same conditions as those which produce an oscillatory instability in Walin's analysis.

2. Analysis

For a fluid which is weakly stratified in both temperature and salt concentration, the equation of state takes the form

$$(\rho_s-\rho^*) = \gamma_c(c_s-c^*)-\gamma_T(T_s-T^*), \tag{1}$$

where ρ, c and T are the density, salt concentration and temperature. γ_c and γ_T are constants, the subscript s denotes properties within the wave and the superscript $*$ background conditions at the origin of the co-ordinate system

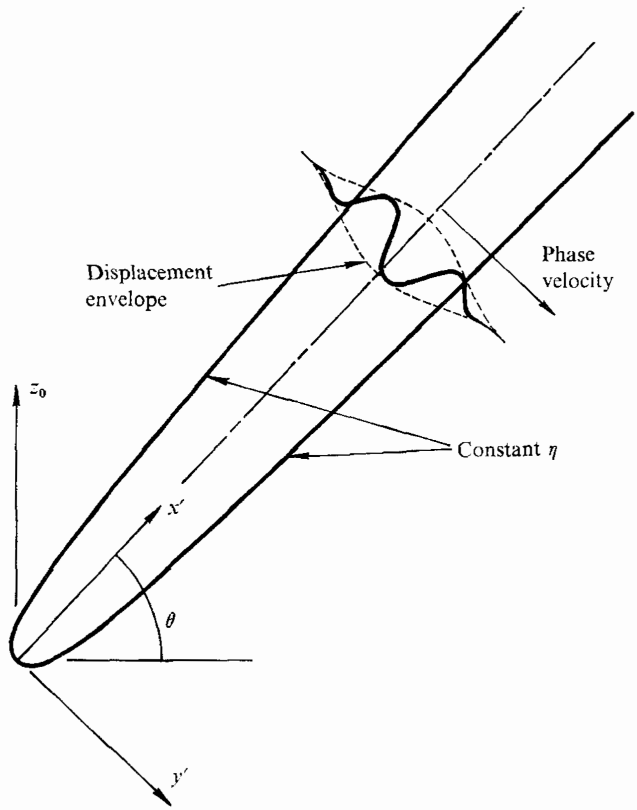

FIGURE 1. The co-ordinate system.

$Ox'y'$ (see figure 1). The co-ordinate system is fixed relative to the background fluid with x' along one arm of the cross and y' in the direction of the phase velocity of the inviscid solution. θ is the angle between the Ox' axis and the horizontal.

Perturbation variables are defined as $\rho' = \rho_s - \rho_0$, $c' = c_s - c_0$ and $T' = T_s - T_0$, where the subscript 0 refers to the unperturbed background conditions. u' and v' are the velocity components and t' is the time. The equations of continuity, momentum and diffusion are written within the Boussinesq approximation as

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{2}$$

$$\frac{Du'}{Dt'} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x'} + \nu_0 \nabla^2 u' - \frac{\rho'}{\rho_0} g \sin\theta, \tag{3}$$

where $$\frac{D}{Dt'} \equiv \frac{\partial}{\partial t'} + u'\frac{\partial}{\partial x'} + v'\frac{\partial}{\partial y'}, \quad \nabla^2 \equiv \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2},$$

$$\frac{Dv'}{Dt'} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y'} + \nu_0 \nabla^2 v' + \frac{\rho'}{\rho_0} g \cos\theta, \tag{4}$$

$$\frac{DT_s}{Dt'} = \chi_T \nabla^2 T_s + \Phi, \tag{5}$$

$$\frac{Dc_s}{Dt'} = \chi_c \nabla^2 c_s. \tag{6}$$

ν is the kinematic viscosity, g is the acceleration due to gravity, Φ is the viscous dissipation and χ_T and χ_c are the heat and salt diffusion coefficients respectively.

The density gradient parameter β is $-\rho_0^{-1}\,(d\rho_0/dz_0)$, where z_0 is measured vertically upwards (see figure 1), and the temperature and salt concentration gradients are given by $dT_0/dz_0 = \beta_T\rho^*/\gamma_T$ and $dc_0/dz_0 = -\beta_c\rho^*/\gamma_c$. The Brunt–Väisälä frequency ω_0 is $(\beta g)^{\frac{1}{2}}$.

Dimensionless variables (undashed) are defined as follows: $x' = x(\beta\sin\theta)^{-1}$, $y' = \alpha y(\beta\sin\theta)^{-1}$, $\rho' = a\rho\rho^*$, $t' = t(\omega_0\sin\theta)^{-1}$, $u' = aug\omega_0^{-1}$, $v' = a\alpha vg\omega_0^{-1}$, $p' = a\alpha p\rho^* g(\beta\tan\theta)^{-1}$, $T' = aT\rho^*\gamma_T^{-1}$, $c' = -ac\rho^*\gamma_c^{-1}$ and $\alpha^3 = \omega_0^3\nu^*\tan\theta\sin\theta/2g^2$. a is an amplitude coefficient which is constant.

Within the Boussinesq approximation we can consider a linear density gradient with $\beta = \beta_T+\beta_c$. As the analysis now follows that in I it will merely be outlined. It is assumed that $a \ll \alpha \ll 1$, so that (1)–(6) reduce to

$$\rho = -(T+c), \tag{7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{8}$$

$$\frac{\partial u}{\partial t} = -\alpha\frac{\partial p}{\partial x}\cot\theta-\rho+2\alpha\cot\theta\frac{\partial^2 u}{\partial y^2}+O(a)+O(\alpha^2), \tag{9}$$

$$\alpha\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}\cot\theta+\rho\cot\theta+O(a)+O(\alpha^2), \tag{10}$$

$$\frac{\partial T}{\partial t}+\frac{\beta_T}{\beta}\{u-\alpha v\cot\theta\} = 2\alpha\frac{\chi_T}{\nu^*}\cot\theta\frac{\partial^2 T}{\partial y^2}+O(a)+O(\alpha^2), \tag{11}$$

$$\frac{\partial c}{\partial t}+\frac{\beta_c}{\beta}\{u-\alpha v\cot\theta\} = 2\alpha\frac{\chi_c}{\nu^*}\cot\theta\frac{\partial^2 c}{\partial y^2}+O(a)+O(\alpha^2). \tag{12}$$

It is assumed that the perturbations have a time dependency e^{-it} and that they can be expanded as $u = u_1+\alpha u_2, \ldots, v = v_1+\alpha v_2 \ldots, p = p_1+\alpha p_2 \ldots$, etc. These expansions are substituted into the above equations and terms of like order are equated. The resulting equation for p_1 is

$$i\frac{\partial p_1}{\partial X} = \frac{\partial^3 p_1}{\partial y^3}, \tag{13}$$

where $X = \kappa x$; $\kappa = [1+(\chi_T\beta_T/\nu^*\beta)+(\chi_c\beta_c/\nu^*\beta)]$.

Equation (13) is of the same form as that in I and the solution is

$$p_1 = \mathrm{Re}\,\{X^{-\frac{1}{3}}f(\eta)\,e^{-it}\},$$

where
$$f = \int_0^\infty \exp(-K^3)\exp(iK\eta)\,dK \quad (\eta = y/X^{\frac{1}{3}}).$$

Thus the Boussinesq form of the solution in I can accommodate the effects of salinity and thermal stratification.

Walin (1964) looked at the stability of disturbances in a viscous fluid stratified by both heat and salt and found that there is an oscillatory mode of instability for some wavenumbers whenever $\kappa < 0$. This is also the limit in the present analysis: no waves of this type exist when $\kappa < 0$. When $\kappa \to 0$ the wave width approaches zero.

If there is no temperature gradient in the fluid then $\beta_c = \beta$ and $\kappa = [1 + (\chi_c/\nu^*)]$. For brine χ_c/ν^* is $O(10^{-3})$ and therefore within the accuracy of the approximations $\kappa = 1$ as in I. If there is no solute gradient in the fluid then $\beta_T = \beta$ and

$$\kappa = [1 + (\chi_T/\nu^*)].$$

For water χ_T/ν^* is $O(10^{-1})$. Thus, if we compare a wave in a thermally stratified fluid with one in a salinity stratified fluid having the same density gradient then the wave width is slightly larger and the velocities decay more rapidly in the thermal wave.

The width of a wave is given by $\eta(x'\nu^*\kappa/2\omega_0 \cos\theta)^{\frac{1}{3}}$ and therefore the weaker the stratification the wider the wave. However a decrease of two orders of magnitude in the density gradient only doubles the wave width at a particular x' position.

Finally, the same approach was used for the vertical wave which develops above and below a body oscillating with the Brunt–Väisälä frequency. As expected, heat conduction and solute diffusion modify the theory of Gordon & Stevenson (1972) and it is found that the dimensionless distance x along the wave is again changed to κx.

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